Hedging in Field Theory Models of the Term Structure

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Abstract

We use path integrals to calculate hedge parameters and efficacy of hedging in a quantum field theory generalization of the Heath, Jarrow and Morton (HJM) term structure model which parsimoniously describes the evolution of imperfectly correlated forward rates. We also calculate, within the model specification, the effectiveness of hedging over finite periods of time. We use empirical estimates for the parameters of the model to show that a low dimensional hedge portfolio is quite effective.

1 Introduction

The first interest models were spot rate models which had only one factor which implied that the prices of all bonds[1] were perfectly correlated. This was observed not to be the case in practice. This observation led Heath, Jarrow and Morton [?] to develop their famous model (heceforth called the HJM model) where the forward rate curve was influenced by more than one factor. This enabled bond prices to have imperfect correlation. However, for an N factor HJM model, this still meant that the movements in the price of N bonds would determine the movements in the prices of all other bonds. This would enable one to hedge any instrument with N bonds within the framework of this model. However, this does not again seem to be the case in practice. In fact, if taken to be exact, a two factor HJM model would seem to imply that one can hedge a thirty year treasury bond with three month and six month bills which is not reasonable. Hence, there has been much interest in developing models which do not have this problem. One possibility is to use an infinite factor HJM model as pointed out in Cohen and Jarrow [?] but it is well known that estimating the parameters of even a two or three factor HJM model from market data is very difficult. In contrast, the estimation of parameters for different field theory models has been discussed in Baaquie and Srikant [?] and is seen to be more effective.

These observations led Kennedy [?], Santa-Clara and Sornette [?] and Goldstein [?] to come up with random field models which allowed imperfect correlations across all the bonds. Baaquie [?] furthered this development by putting all these models into a field theory framework which allows for the use of a large body of theoretical and computational methods developed in physics to be applied to this problem.

2 A Brief Summary of the HJM and Field Theory Models

In the HJM model the forward rates are given by

$$f(t,x) = f(t_0,x) + \int_{t_0}^{t} dt' \alpha(t',x) + \sum_{i=1}^{K} \int_{t_0}^{t} dt' \sigma_i(t',x) dW_i(t') \tag{1}$$

where W_i are independent Wiener processes. We can also write this as

$$\frac{\partial f(t,x)}{\partial t} = \alpha(t,x) + \sum_{i=1}^{K} \sigma_i(t,x)\eta_i(t) \tag{2}$$

[1]In this paper, we only use zero coupon bonds, hence all references to bonds are to zero coupon bonds

where η_i represent independent white noises. The action functional, is

$$S[W] = -\frac{1}{2}\sum_{i=1}^{K}\int dt\eta_i^2(t) \tag{3}$$

We can use this action to calculate the generating functional which is

$$\begin{aligned} Z[j, t_1, t_2] &= \int \mathcal{D}W e^{\sum_{i=1}^{K}\int_{t_1}^{t_2} dt j_i(t)W_i(t)} e^{S_0[W,t_1,t_2]} \\ &= e^{\frac{1}{2}\sum_{i=1}^{K}\int_{t_1}^{t_2} dt j_i^2(t)} \end{aligned} \tag{4}$$

We now review Baaquie's field theory model presented in [?] with constant rigidity. Baaquie proposed that the forward rates being driven by white noise processes in (2) be replaced by considering the forward rates itself to be a quantum field. To simplify notation, we write the evolution equation in terms of the velocity quantum field $A(t, x)$, and which yields

$$f(t, x) = f(t_0, x) + \int_{t_0}^{t} dt'\alpha(t', x) + \sum_{i=1}^{K}\int_{t_0}^{t} dt'\sigma_i(t', x)A_i(t', x) \tag{5}$$

or

$$\frac{\partial f(t, x)}{\partial t} = \alpha(t, x) + \sum_{i=1}^{K}\sigma_i(t, x)A_i(t, x) \tag{6}$$

The main extension to HJM is that A depends on x as well as t unlike W which only depends on t.

While we can put in many fields A_i, it was shown in Baaquie and Srikant [?] that the extra generality brought into the process due to the extra argument x makes one field sufficient. Hence, in future, we will drop the subscript for A.

Baaquie further proposed that the field A has the free (Gaussian) free field action functional [?]

$$S = -\frac{1}{2}\int_{t_0}^{\infty} dt \int_{t}^{t+T_{FR}} dx \left(A^2 + \frac{1}{\mu^2}\left(\frac{\partial A}{\partial x}\right)^2\right) \tag{7}$$

with Neumann boundary conditions imposed at $x = t$ and $x = t+T_{FR}$. This makes the action equivalent (after an integration by parts where the surface term vanishes) to

$$S = -\frac{1}{2}\int_{t_0}^{\infty} dt \int_{t}^{t+T_{FR}} dx A(t, x)\left(1 - \frac{1}{\mu^2}\frac{\partial^2}{\partial x^2}\right) A(t, x) \tag{8}$$

This action has the partition function

$$Z[j] = \exp\left(\int_{0}^{t_1} dt \int_{t}^{t+T_{FR}} dx dx' j(t, x)D(x - t, x' - t)j(t, x')\right) \tag{9}$$

with

$$\begin{aligned} D(\theta, \theta'; T_{FR}) &= \mu\frac{\cosh\mu\,(T_{FR} - |\theta - \theta'|) + \cosh\mu\,(T_{FR} - (\theta + \theta'))}{2\sinh\mu T_{FR}} \\ &= D(\theta', \theta; T_{FR}) \quad : \quad \text{Symmetric Function of } \theta, \theta' \end{aligned} \tag{10}$$

where $\theta = x - t$ and $\theta' = x' - t$. We can calculate expectations and correlations using this partition function. Note that due to the boundary conditions imposed, the inverse of the differential operator D actually depends only the difference $x - t$. The above action represents a Gaussian random field with covariance structure D. In [?], a different form was found as the boundary conditions used were Dirichlet with the endpoints integrated over. This boundary condition is in fact equivalent to the Neumann condition which leads to the much simpler propagator above. In the limit $T_{FR} \to \infty$ which we will usually take, the propagator takes the simple form $\mu e^{-\mu\theta_>}\cosh\mu\theta_<$ where $\theta_>$ and $\theta_<$ stand for $\max(\theta, \theta')$ and $\min(\theta, \theta')$ respectively.

When $\mu \to 0$, this model should go over to the HJM model. This is indeed seen to be the case as it is seen that $\lim_{\mu \to 0} D(\theta, \theta'; T_{FR}) = \frac{1}{T_{FR}}$. The extra factor of T_{FR} is irrelevant as it is due to the freedom we have in scaling σ and D. The σ we use for the different models are only comparable after D is normalized[2]. On normalization, the propagator for both the HJM model and field theory model in the limit $\mu \to 0$ is one showing that the two models are equivalent in this limit.

3 Hedging

The main aim of hedging is to reduce one's exposure to risk. There are many ways to define risk [**?**]. For bonds, the main risks are changes in interest rates and the risk of default. In this paper, we are only dealing with default-free bonds so that the only source of risk is the change in interest rates.

For the purposes of this paper, we define risk to be the standard deviation or variance of final value. Hence, when we hedge a certain instrument, we are trying to create a portfolio of the hedged and hedging instruments which minimizes the overall variance of the portfolio. In the case of a N-factor HJM model, perfect hedging (i.e., a zero portfolio variance) is achievable once any N independent hedging instruments are used. However, the difficulties introduced by the infinite number of factors in the field theory models has resulted in their being very little literature on this important subject, a notable exception being the measure valued trading strategy developed in Björk, Kabanov and Runggaldier[**?**].

In the fourth section of this paper, we will consider instantaneous hedging which is important for theoretical purposes. We will calculate the maximum reduction in variance for a finite number of hedging instruments and the hedge ratios (the amount of hedging instrument that requires to be used) that result in this maximization. This will show us how well the model can be approximated by a finite number of factors. We will then use the constant rigidity model fitted with empirical data to estimate the reduction in the variance of an optimally hedged portfolio as the number of hedging instruments are increased. We will see that a relatively small number of hedging instruments gives good results. We will also show that the results reduced to well known textbook ones as in Jarrow and Turnbull[**?**] when we go to the degenerate case of one-factor HJM model where all the forward rate innovations are perfectly correlated. We will also perform the same calculations using the propagator estimated from empirical data.

In the third section, we will consider finite time hedging which is important in practice. This is because continuous hedging cannot be done in practice due to the presence of transaction costs. We will see how the hedging performance found in the second section changes as the time between rebalancings is increased. The entire analysis here is to investigate how portfolios of bonds behave in such models.

4 Instantaneous Hedging

In instantaneous hedging, we are considering a hedging portfolio which is rebalanced continuously in time. Hence, we are only considered with the instantaneous variance of the portfolio. This can be calculated for an arbitrary portfolio by using the fact that the covariance of the innovations in the forward rates is given by

$$\sigma(\theta) D(\theta, \theta') \sigma(\theta') \tag{11}$$

in the field theory model. We will only present the hedging of zero coupon bonds in this section though it will be seen that the results can be easily extended to other instruments. In the first subsection, we will present the theoretical derivation of the hedge ratios and reduced variance for the hedging of a zero coupon bond with other zero coupon bonds. In the second subsection, we use the empirically fitted σ and $D(\mu)$ (from (10)) discussed in Baaquie and Srikant [**?**] for the constant rigidity action as well as the non-parametric estimate for σ and D which is directly obtained from the market correlation matrix of the innovations in forward rates to calculate the semi-empirical reduction in variance. In the third and fourth subsections, we will carry out similar calculations when hedging zero coupon bonds with futures on zero coupon bonds. This is much more realistic in practice as hedging with futures is relatively cheap.

[2]This freedom exists since we can always make the transformation $\sigma(\theta) \sim \eta(\theta)\sigma(\theta)$ and $D(\theta, \theta') \sim D(\theta, \theta')/(\eta(\theta)\eta(\theta'))$ without affecting any result

4.1 Hedging bonds with other bonds

We now consider the hedging of one bond maturing at T with N other bonds maturing at $T_i, 1 \leq i \leq N$. If one of the $T_i = T$, then the solution is trivial since it is the same bond. The hedge is then just to short the same bond giving us a zero portfolio with obviously zero variance. Since this solution is uninteresting, we assume that $T_i \neq T\ \forall i$. The hedged portfolio $\Pi(t)$ can then be represented as

$$\Pi(t) = P(t,T) + \sum_{i=1}^{N} \Delta_i P(t,T_i)$$

where Δ_i denotes the amount of the i^{th} bond $P(t,T_i)$ included in the hedged portfolio. Note the value of bonds $P(t,T)$ and $P(t,T_i)$ are determined by observing their market values at time t. It is the instantaneous *change* in the portfolio value that is stochastic. Therefore, the variance of this change is computed to ascertain the efficacy of the hedge portfolio.

We first consider the variance of the value of an individual bond in the field theory model. The definition $P(t,T) = \exp\left(-\int_t^T dx f(t,x)\right)$ for zero coupon bond prices implies that

$$\begin{aligned}
\frac{dP(t,T)}{P(t,T)} &= f(t,t)dt - \int_0^{T-t} d\theta df(t,\theta) \\
&= \left(r(t) - \int_0^{T-t} d\theta \alpha(\theta) - \int_0^{T-t} d\theta \sigma(\theta) A(t,\theta) \right) dt
\end{aligned}$$

and $E\left[\frac{dP(t,T)}{P(t,t)}\right] = \left(r(t) - \int_0^{T-t} d\theta\alpha(\theta)\right) dt$ since $E[A(t,\theta)] = 0$. Therefore

$$\frac{dP(t,T)}{P(t,T)} - E\left[\frac{dP(t,T)}{P(t,T)}\right] = -dt \int_0^{T-t} d\theta \sigma(\theta) A(t,\theta) \tag{12}$$

Squaring this expression and invoking the result that $E[A(t,\theta)A(t,\theta')] = \delta(0)D(\theta,\theta';T_{FR}) = \frac{D(\theta,\theta';T_{FR})}{dt}$ results in the instantaneous bond price variance

$$Var[dP(t,T)] = dtP^2(t,T) \int_0^{T-t} d\theta \int_0^{T-t} d\theta' \sigma(\theta) D(\theta,\theta';T_{FR}) \sigma(\theta') \tag{13}$$

As an intermediate step, the instantaneous variance of a bond portfolio is considered. For a portfolio of bonds, $\hat{\Pi}(t) = \sum_{i=1}^{N} \Delta_i P(t,T_i)$, the following results follow directly

$$d\hat{\Pi}(t) - E[d\hat{\Pi}(t)] = -dt \sum_{i=1}^{N} \Delta_i P(t,T_i) \int_0^{T_i - t} d\theta \sigma(\theta) A(t,\theta) \tag{14}$$

and

$$Var[d\hat{\Pi}(t)] = dt \sum_{i=1}^{N} \sum_{j=1}^{N} \Delta_i \Delta_j P(t,T_i) P(t,T_j) \int_0^{T_i - t} d\theta \int_0^{T_j - t} d\theta' \sigma(\theta) D(\theta,\theta';T_{FR}) \sigma(\theta') \tag{15}$$

The (residual) variance of the hedged portfolio

$$\Pi(t) = P(t,T) + \sum_{i=1}^{N} \Delta_i P(t,T_i) \tag{16}$$

may now be computed in a straightforward manner. For notational simplicity, the bonds $P(t,T_i)$ (being used to hedge the original bond) and $P(t,T)$ are denoted P_i and P respectively. Equation (15) implies

the hedged portfolio's variance equals the final result shown below

$$\begin{aligned}
& P^2 \int_0^{T-t} d\theta \int_0^{T-t} d\theta' \sigma(\theta)\sigma(\theta') D(\theta,\theta';T_{FR}) \\
& +2P \sum_{i=1}^{N} \Delta_i P_i \int_0^{T-t} d\theta \int_0^{T_i-t} d\theta' \sigma(\theta)\sigma(\theta') D(\theta,\theta';T_{FR}) \\
& + \sum_{i=1}^{N}\sum_{j=1}^{N} \Delta_i \Delta_j P_i P_j \int_0^{T_i-t} d\theta \int_0^{T_j-t} d\theta' \sigma(\theta) D(\theta,\theta';T_{FR})\sigma(\theta')
\end{aligned} \tag{17}$$

Note that the residual variance depends on the correlation structure of the innovation in forward rates described by the propagator D. Ultimately, the effectiveness of the hedged portfolio is an empirical question since perfect hedging is not possible without shorting the original bond. This empirical question is addressed in the next subsection where the propagator calibrated to market data is used to calculate the effectiveness. Minimizing the residual variance in equation (17) with respect to the hedge parameters Δ_i is an application of standard calculus. We introduce the following notation for simplicity.

$$\begin{aligned}
L_i &= PP_i \int_0^{T-t} d\theta \int_0^{T_i-t} d\theta' \sigma(\theta)\sigma(\theta') D(\theta,\theta';T_{FR}) \\
M_{ij} &= P_i P_j \int_0^{T_i-t} d\theta \int_0^{T_j-t} d\theta' \sigma(\theta)\sigma(\theta') D(\theta,\theta';T_{FR})
\end{aligned}$$

L_i is the covariance between the innovations in the hedged bond and the ith hedging bond and M_{ij} is the covariance between the innovations of the ith and jth hedging bond.

The above definitions allow the residual variance in equation (17) to be succinctly expressed as

$$P^2 \int_0^{T-t} d\theta \int_0^{T-t} d\theta' \sigma(\theta)\sigma(\theta') D(\theta,\theta';T_{FR}) + 2\sum_{i=1}^{N} \Delta_i L_i + \sum_{i=1}^{N}\sum_{j=1}^{N} \Delta_i \Delta_j M_{ij} \tag{18}$$

The hedge parameters in the field theory model can now be evaluated using basic calculus and linear algebra to obtain

$$\Delta_i = -\sum_{j=1}^{N} L_j M_{ij}^{-1} \tag{19}$$

and represent the optimal amounts of $P(t,T_i)$ to include in the hedge portfolio when hedging $P(t,T)$.

Putting the result into (17), we see that the variance of the hedged portfolio equals

$$V = P^2 \int_0^{T-t} d\theta \int_0^{T-t} d\theta' \sigma(\theta)\sigma(\theta') D(\theta,\theta';T_{FR}) - \sum_{i=1}^{N}\sum_{j=1}^{N} L_i M_{ij}^{-1} L_j \tag{20}$$

which declines monotonically as N increases.

The residual variance enables the effectiveness of the hedged portfolio to be evaluated. Therefore, this result is the basis for studying the impact of including different bonds in the hedged portfolio as illustrated in the next subsection. For $N=1$, the hedge parameter reduces to

$$\Delta_1 = -\frac{P}{P_1}\left(\frac{\int_0^{T-t} d\theta \int_0^{T_1-t} d\theta' \sigma(\theta)\sigma(\theta') D(\theta,\theta';T_{FR})}{\int_0^{T_1-t} d\theta \int_0^{T_1-t} d\theta' \sigma(\theta)\sigma(\theta') D(\theta,\theta';T_{FR})}\right) \tag{21}$$

To obtain the HJM limit, we let the propagator equal one. The hedge parameter in equation (21) then reduces to

$$\Delta_1 = -\frac{P}{P_1}\left(\frac{\int_0^{T-t} d\theta \int_0^{T_1-t} d\theta' \sigma(\theta)\sigma(\theta')}{\left(\int_0^{T_1-t} d\theta\sigma(\theta)\right)^2}\right) = -\frac{P}{P_1}\left(\frac{\int_0^{T-t} d\theta\sigma(\theta)}{\int_0^{T_1-t} d\theta\sigma(\theta)}\right) \tag{22}$$

The popular exponential volatility function[3] $\sigma(\theta) = \sigma e^{-\lambda\theta}$ allows a comparison between our field theory solutions and previous research. Under the assumption of exponential volatility, equation (22) becomes

$$\Delta_1 = -\frac{P}{P_1}\left(\frac{1 - e^{-\lambda(T-t)}}{1 - e^{-\lambda(T_1-t)}}\right) \tag{23}$$

Equation (23) coincides with the ratio of hedge parameters found as equation 16.13 of Jarrow and Turnbull [?]. In terms of their notation

$$\Delta_1 = -\frac{P(t,T)}{P(t,T_1)}\left(\frac{X(t,T)}{X(t,T_1)}\right) \tag{24}$$

For emphasis, the following equation holds in a one factor HJM model[4]

$$\frac{\partial\left[P(t,T) + \Delta_1 P(t,T_1)\right]}{\partial r(t)} = 0 \tag{25}$$

which is verified using equation (24) and results found on pages 494-495 of Jarrow and Turnbull [?]

$$\begin{aligned}\frac{\partial\left[P(t,T) + \Delta_1 P(t,T_1)\right]}{\partial r(t)} &= -P(t,T)X(t,T) - \Delta_1 P(t,T_1)X(t,T_1) \\ &= -P(t,T)X(t,T) + P(t,T)X(t,T) = 0\end{aligned}$$

When $T_1 = T$, the hedge parameter equals minus one. Economically, this fact states that the best strategy to hedge a bond is to short a bond of the same maturity. This trivial approach reduces the residual variance in equation (18) to zero as $\Delta_1 = -1$ and $P = P_1$ implies $L_1 = M_{11}$. Empirical results for nontrivial hedging strategies are found in the next subsection where the calibrated propagator is used.

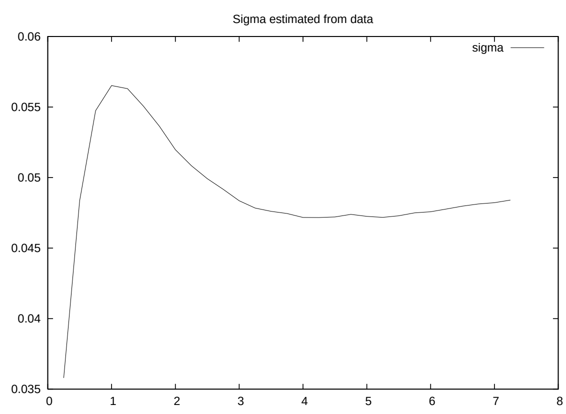

Figure 1: Implied volatility function (unnormalized) for constant rigidity model using market data

4.2 Semi-empirical results : Constant Rigidity model

The empirical estimation of parameters for the field theory model was explained in detail in Baaquie and Srikant [?]. For this subsection, we use the function σ estimated for the constant rigidity model from market data. The function σ is plotted in figure 1.

This approach preserves the closed form solutions for hedge parameters and futures contracts illustrated in the previous subsection. However, the original finite factor HJM model cannot accommodate an empirically determined propagator since it is automatically fixed once the HJM volatility functions are specified. Later in this subsection, we will see how the empirical propagator modifies the results

[3] This volatility function is commonly used as it lets the spot rate $r(t)$ follow a Markov process. See [?].

[4] Note that this result depends on the fact that the spot rate $r(t)$ is Markovian and therefore only applies to either a constant or exponential volatility function.

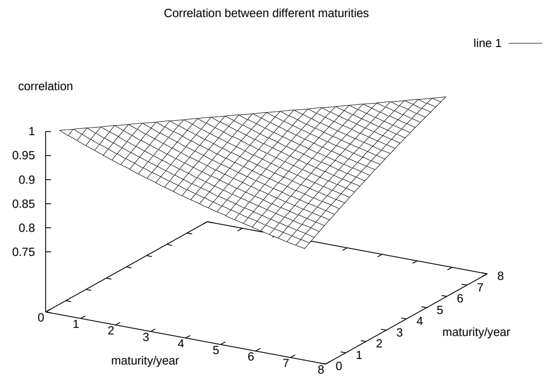

Figure 2: Propagator Implied by the constant rigidity field theory model with $\mu = 0.06/\text{year}$

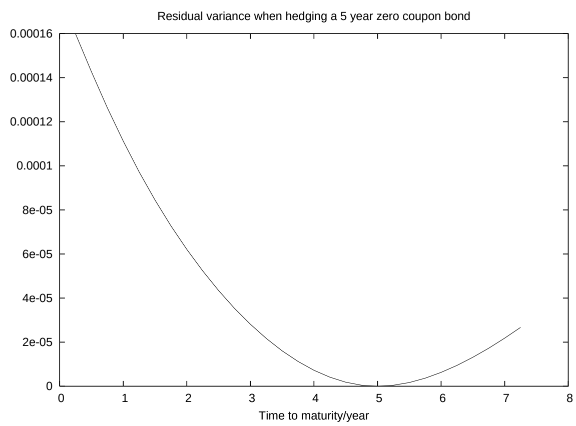

Figure 3: Residual variance for five year bond versus bond maturity used to hedge

of this subsection. The implied propagator for the empirically fitted value of $0.06/yr$ for μ is shown in figure 2.

The reduction in variance achievable by hedging a five year bond with other bonds is the focus of this subsection. We take the current forward rate curve to be flat and equal to 5% throughout. The initial forward rate curve does not affect any of the qualitative results. The results can also be easily extended to other bonds. The residual variances for one and two bond hedged portfolios are shown in figures 3 and 4. The calculation of the integrals involved was done using simple trapezoidal integration as the data is not exceptionally accurate in the first place. Secondly and more importantly, the errors involved will largely cancel themselves out, hence the difference in the variances is still quite accurate. For example, in figures 3 and 4, we can see that in the case of perfect hedging, we get exactly zero residual variance which shows that the errors tend to cancel. The parabolic nature of the residual variance is because μ is constant. A more complicated function would produce residual variances that do not deviate monotonically as the maturity of the underlying and the hedge portfolio increases although the graphs appeal to our economic intuition which suggests that correlation between forward rates decreases monotonically as the distance between them increases as shown in figure 2. Observe that the residual variance drops to zero when the same bond is used to hedge itself, eliminating the original position in the process. The corresponding hedge ratios are shown in figure 5.

It is also interesting to note that hedging by two bonds, even very closely spaced ones, seems to be bring significant additional benefits. This can be seen in figure 4 where the diagonal $\theta = \theta'$ represents hedging by one bond. The residual variance there is higher than the nearby points in a discontinuous manner.

We now present the results for the actual propagator found from the data which is graphed in figure 6. The residual variance when a five year bond is hedged with one and two bonds bond is shown in figures 8 and 9. We can see from figure 9 that, when the market propagator is used, the advantage of using more

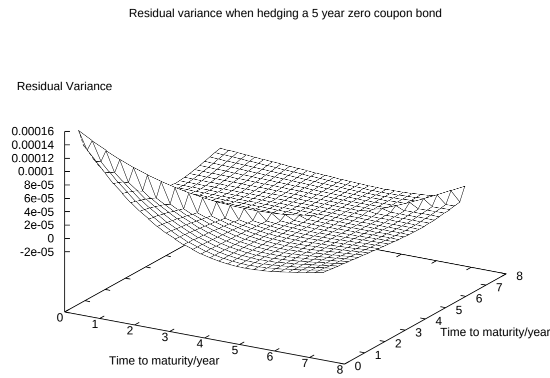

Figure 4: Residual variance for five year bond versus two bond maturities used to hedge

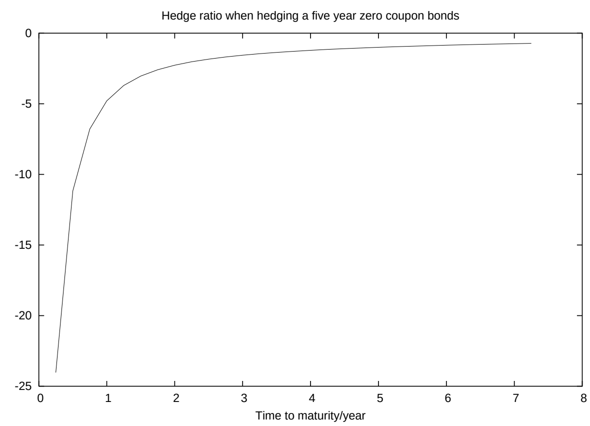

Figure 5: Hedge ratios for five year bond

than one bond to hedge is significantly higher. This is because of the nature of the correlation structure in figure 6. We see that the correlations of innovations of nearby forward rates of higher maturity is significantly higher in the market propagator, making hedging with more than one bond more useful. This is even more pronounced when hedging a short maturity bond with longer maturity ones. We can see this from figure 10 which shows the residual variance when a one year bond is hedged with two bonds where the calculation is done using the empirical propagator. The effect of this higher correlation among forward rates of higher maturity can also be seen in figure 9 where the residual variance rises much more slowly when the hedging bonds are chosen to be of higher maturity than the hedged bond.

4.3 Hedging with futures

We can carry out an analysis very similar to subsection 4.1 to find the optimal hedge ratios when hedging a bond with futures contracts on the same or other bonds. In this case, there is no trivial solution to the hedging problem as when bonds were hedged with other bonds. Further, since this method of hedging is much more practical in reality, the results will be more interesting. Proceeding as in subsection 4.1, we compute the appropriate hedge parameters for futures contracts. The futures price $\mathcal{F}(t, t_*, T)$ in terms of the forward price $\frac{P(t,T)}{P(t,t_*)} = e^{-\int_{t_*-t}^{T-t} d\theta f(t,\theta)}$ and the *deterministic* quantity $\Omega_{\mathcal{F}}(t, t_*, T)$ which is given by [**?**]

$$\Omega_{\mathcal{F}}(t_0, t_*, T) = -\sum_{i=1}^{N} \int_{t_0}^{t_*} dt \int_0^{t_*-t} d\theta \sigma_i(t,\theta) \int_{t_*-t}^{T-t} d\theta' \sigma_i(t,\theta') \qquad (26)$$

The dynamics of the futures price $d\mathcal{F}(t, t_*, T)$ is thus given by

$$\frac{d\mathcal{F}(t, t_*, T)}{\mathcal{F}(t, t_*, T)} = d\Omega_{\mathcal{F}}(t, t_*, T) - \int_{t_*-t}^{T-t} d\theta df(t,\theta) \qquad (27)$$

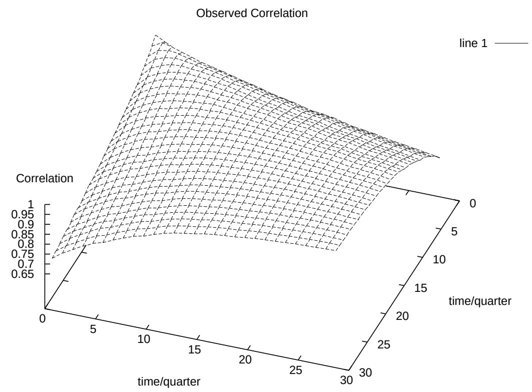

Figure 6: The propagator implied by the market data

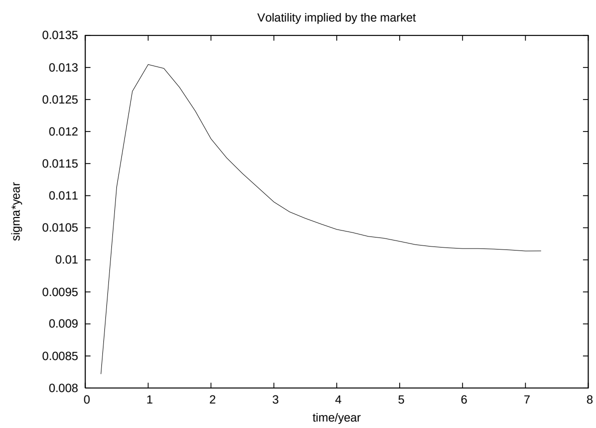

Figure 7: The volatility implied by the market data when using the empirical propagator

which implies

$$\frac{d\mathcal{F}(t,t_*,T) - E[d\mathcal{F}(t,t_*,T)]}{\mathcal{F}(t,t_*,T)} = -dt \int_{t_*-t}^{T-t} d\theta \sigma(\theta) A(t,\theta) \tag{28}$$

Squaring both sides as before leads to the instantaneous variance of the futures price

$$Var[d\mathcal{F}(t,t_*,T)] = dt\mathcal{F}^2(t,t_*,T) \int_{t_*-t}^{T-t} d\theta \int_{t_*-t}^{T-t} d\theta' \sigma(\theta) D(\theta,\theta') \sigma(\theta') \tag{29}$$

Let $\mathcal{F}_i$ denote the futures price $\mathcal{F}(t,t_*,T_i)$ of a contract expiring at time t_* on a zero coupon bond maturing at time T_i. The hedged portfolio in terms of the futures contract is given by

$$\Pi(t) = P + \sum_{i=1}^{N} \Delta_i \mathcal{F}_i \tag{30}$$

where $\mathcal{F}_i$ represent observed market prices. For notational simplicity, define the following terms

$$\begin{aligned} L_i &= P\mathcal{F}_i \int_{t_*-t}^{T_i-t} d\theta \int_0^{T-t} d\theta' \sigma(\theta) D(\theta,\theta';T_{FR}) \sigma(\theta') \\ M_{ij} &= \mathcal{F}_i \mathcal{F}_j \int_{t_*-t}^{T_i-t} d\theta \int_{t_*-t}^{T_j-t} d\theta' \sigma(\theta) D(\theta,\theta';T_{FR}) \sigma(\theta') \end{aligned}$$

The hedge parameters and the residual variance when futures contracts are used as the underlying hedging instruments have identical expressions to those in (19) and (20) but are based on the new definitions of L_i and M_{ij} above. Computations parallel those in section 4.1.

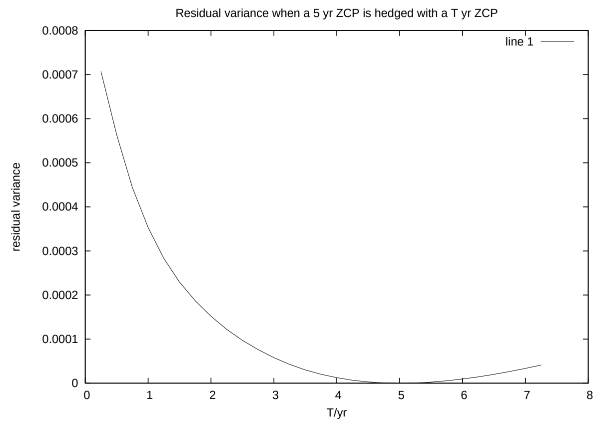

Figure 8: The residual variance when a five year bond is hedged with one bond

To explicitly state the results, the hedge parameters for a futures contract that expires at time t_* on a zero coupon bond that matures at time T_i equals

$$\Delta_i = -\sum_{j=1}^{N} L_j M_{ij}^{-1}$$

while the variance of the hedged portfolio equals

$$V = P^2 \int_0^{T-t} d\theta \int_0^{T-t} d\theta' \sigma(\theta)\sigma(\theta') D(\theta, \theta'; T_{FR}) - \sum_{i=1}^{N}\sum_{j=1}^{N} L_i M_{ij}^{-1} L_j$$

for L_i and M_{ij} as defined above.

4.4 Semi-empirical results for hedging with futures

We first present results for the propagator fitted for the constant rigidity model as for the bonds. The initial forward rate curve is again taken to be flat and equal to 5%. We also fix the expiry of the futures contracts to be at one year from the present. This is a long enough time to clearly show the effect of the expiry time as well as short enough to make practical sense as long term futures contracts are illiquid and unsuitable for hedging purposes.

The calculations were done using simple trapezoidal integration as explained previously. This is sufficient for our purposes as the fitted values for σ and D shown in figures 7 and 6 are reasonably but not exceptionally accurate and we are more interested in the qualitative behaviour of the residual variance and hedge parameters.

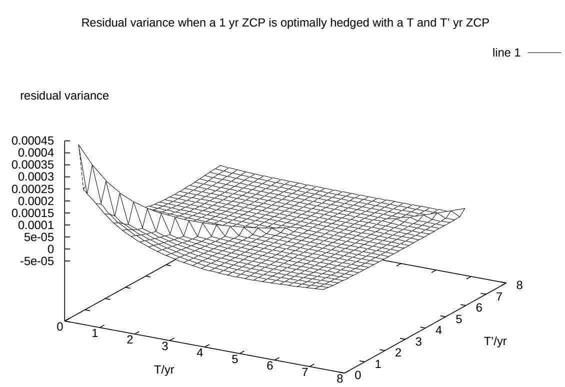

Figure 9: The residual variance when a five year bond is hedged with two bonds

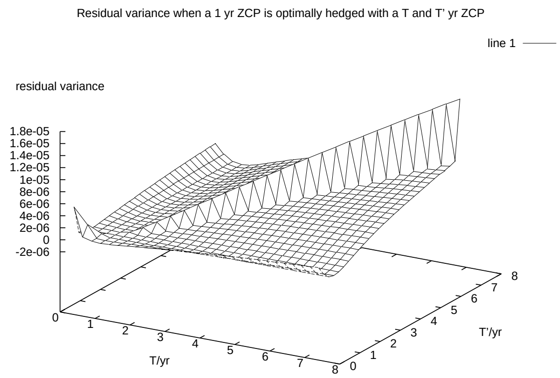

Figure 10: The residual variance when a one year bond is hedged with two bonds

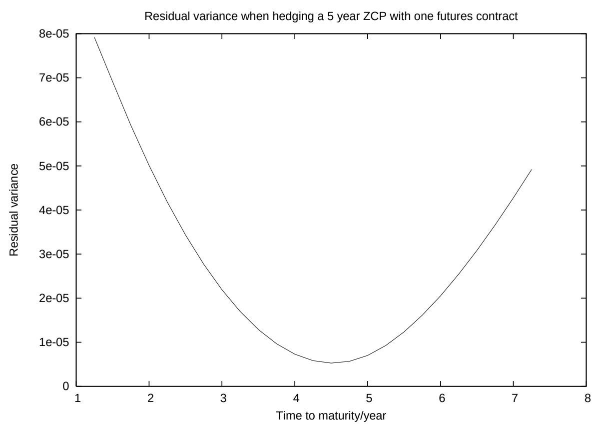

Figure 11: Residual variance for a five year bond hedged with a one year futures contract on a T maturity bond

The residual variance achieved when hedging a five year bond with one futures contract is shown in figure 11. The optimal hedge ratios and the resulting residual variances when hedging with two and three futures are shown in table 1. These were obtained by systematically tabulating all possible combinations of bonds with intervals of three months in the maturity direction, tabulating the residual variance for each and finding the best combination.

Firstly, we note that the hedging is very effective even when one futures contract is used reducing the variance by a factor of over three hundred. Secondly, we also note that the most effective hedging is *not* obtained by shorting the futures corresponding to the same bond but one with a slightly lower maturity. This is due to the correlation structure of the forward rates. However, when two futures contracts are used, we see that one of the optimal contracts is the future on the same bond as well as a very short maturity futures which is probably due to the short end of the forward rate curve which does influence the bond but not the futures. Since the shortest maturity futures contract is probably likely to have the highest correlation with this part of the forward rate curve, it seems reasonable to select this futures contract to balance the effect on the bond from this part of the forward rate curve. This is indeed the case as seen in table 1. We also note that there is very little extra improvement as we use more than two futures.

We now present the same results using the empirical propagator directly. The residual variance when one futures contract is used for hedging is shown in figure 12. The optimal hedging futures, hedge ratios and residual variances are shown in table 2. We see that for the actual propagator, the optimal hedging futures are even farther from the actual underlying bond when compared to the optimal values using the fitted propagator.

Number	Futures Contracts (Hedge Ratio)	Residual Variance
0	none	1.82×10^{-3}
1	4.5 years (-1.288)	5.29×10^{-6}
2	5 years (-0.9347), 1.25 years (-2.72497)	1.58×10^{-6}
3	5 years (-0.95875), 1.5 years (1.45535), 1.25 years (-5.35547)	1.44×10^{-6}

Table 1: Residual variance and hedge ratios for a five year bond hedged with one year futures contracts.

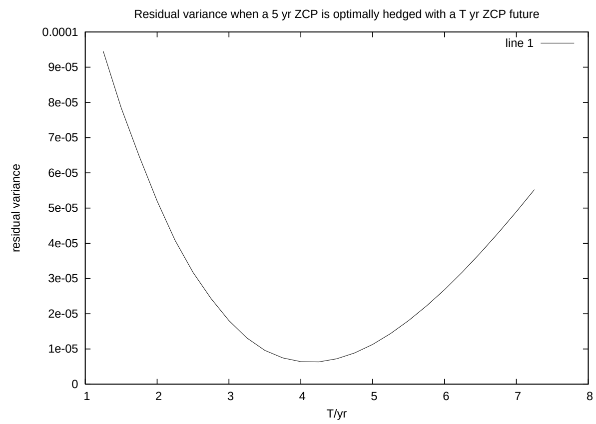

Figure 12: Residual variance when a five year bond is hedged with a one year futures contract on a T maturity bond

5 Finite time hedging

The case of finite time hedging is considerably more complicated. We will only do the hedging of bonds with other bonds as the calculations for minimizing variance can be done exactly. We will not do hedging of bonds with futures even though this can also be solved exactly for minimizing the variance as it does not add much extra insight for finite time. To see this, consider hedging with a futures contract on a zero coupon bond of duration T that matures at the same as the hedging horizon. This gives exactly the same result as hedging with a bond of the same maturity T. Therefore, we gain nothing by carrying out that calculation.

The following calculation proceeds efficiently because of the use of path integral techniques which are very useful for such problems. To be able to optimally hedge bonds with other bonds in the sense of having a minimal residual variance, we need to the covariance between the final values of bonds of different maturities. To calculate this covariance, we will first find the joint probability density function for N bonds at the hedging horizon. Let us denote the initial time by 0, the hedging horizon by t_1 and the maturities of the bonds by T_i. Making use of (9) we obtain the joint distribution of the quantities $G_i = \int_{t_1}^{T_i} dx(f(t_1,x) - f(0,x))$[5] which represent the logarithms of the ratios of final value of the bonds to the value of their forward prices for the final time at the initial time. In other words

$$G_i = \ln\left(\frac{P(t_1,T)P(t_0,t_1)}{P(t_0,T)}\right) = \ln\left(\frac{P(t_1,T)}{F(t_0,t_1,T)}\right) \tag{31}$$

The calculation proceeds as follows

$$\begin{aligned}&\langle\prod_{j=1}^{N}\delta(\int_{t_1}^{T_j} dx(f(t_1,x) - f(0,x)) - G_j)\rangle\\&= \int dp_j \mathcal{D}A \exp\left(i\sum_{j=1}^{N} p_j\left(\int_0^{t_1} dt\int_{t_1}^{T_j} dx\alpha(t,x) + \int_0^{t_1} dt\int_{t_1}^{T_j} dx\sigma(t,x)A(t,x) - G_j\right)\right)\end{aligned} \tag{32}$$

[5]Due to the definition of this quantity, it is easier to carry out the calculations for the finite case with the x rather than the θ variable, hence we use this variable in this section

Number	Futures Contracts (Hedge Ratio)	Residual Variance
0	None	1.74×10^{-3}
1	4.25 years (-0.984)	6.34×10^{-6}
2	1.25 years (-3.84577), 5.5 years (-0.76005)	2.26×10^{-6}
3	1.25 years (-8.60248), 1.5 years (2.84177), 5.25 (-0.85915)	1.95×10^{-6}

Table 2: Residual variance and hedge ratios for a five year bond hedged with one year futures contracts.

which, on applying (9) becomes

$$\begin{aligned}\int dp_j \exp\Bigg(&-\frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N} p_j p_k \int_0^{t_1} dt \int_{t_1}^{T_j} dx \int_{t_1}^{T_k} dx' \sigma(t,x) D(x-t,x'-t)\sigma(t,x') \\ &+i\sum_{j=1}^{N} p_j \left(\int_0^{t_1} dt \int_{t_1}^{T_j} dx \alpha(t,x) - G_j\right)\Bigg)\end{aligned} \tag{33}$$

Performing the Gaussian integrations, we obtain the joint probability distribution given by

$$(2\pi)^{-n/2}(\det B)^{-1/2} \exp\left(-\frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N}(G_j - m_j)B^{-1}_{jk}(G_k - m_k)\right) \tag{34}$$

where B is the matrix whose elements B_{ij} are given by

$$B_{ij} = \int_0^{t_1} dt \int_{t_1}^{T_i} dx \int_{t_1}^{T_j} dx' \sigma(t,x) D(x-t,x'-t)\sigma(t,x') \tag{35}$$

and m_i is given by

$$m_i = \int_0^{t_1} dt \int_{t_1}^{T_i} dx \alpha(t,x) \tag{36}$$

Hence, the quantities G_i follow a multivariate Gaussian distribution with covariance matrix B_{ij} and mean m_i.

Having found the joint distribution of G_i, we can find the covariance of the final bond prices by tabulating the expectations of each of the bonds and the expectation of their products. The final bond price is given by $P(t_1,T_i) = F(0,t_1,T_i)e^{G_i}$ in terms of G_i. Hence, the expectation of this quantity is given by

$$(2\pi)^{-N/2}(\det B)^{-1/2} \int dG_i F(0,t_1,T_i) e^{G_i} \exp\left(-\frac{1}{2}(G-m)^T B^{-1}_{ij}(G-m)\right) \tag{37}$$

which gives $\mathcal{F}(0,t_1,T_i)$ as it must since the expectation of the future bond price is the futures price. The expectation of the products of the prices of two bonds $\langle P(t_1,T_i)P(t_1,T_j)\rangle$ is given by

$$(2\pi)^{-N/2}(\det B)^{-1/2} \int dG_i dG_j F(0,t_1,T_i)F(0,t_1,T_j) e^{G_i+G_j} \exp\left(-\frac{1}{2}H^T B^{-1} H\right) \tag{38}$$

where H stands for the vector $G-m$. On evaluation, this gives the result

$$\mathcal{F}(0,t_1,T_i)\mathcal{F}(0,t_1,T_j) \exp\left(\int_0^{t_1} dt \int_{t_1}^{T_i} dx \int_{t_1}^{T_j} dx' \sigma(t,x) D(x-t,x'-t)\sigma(t,x')\right) \tag{39}$$

We now consider the behaviour of the portfolio

$$P(t,T) + \sum_{i=1}^{N} \Delta_i P(t,T_i) \tag{40}$$

The covariance between the prices $P(t_1, T_i)$ and $P(t_i, T_j)$ is given by

$$\begin{aligned} M_{ij} =& \mathcal{F}(0, t_1, T_i)\mathcal{F}(0, t_1, T_j) \\ & \left(\exp\left(\int_0^{t_1} dt \int_{t_1}^{T_i} dx \int_{t_1}^{T_j} dx' \sigma(t, x) D(x - t, x' - t) \sigma(t, x') \right) - 1 \right) \end{aligned} \tag{41}$$

and the covariance between the hedged bond of maturity T and the hedged bonds is given by

$$\begin{aligned} L_i =& \mathcal{F}(0, t_1, T)\mathcal{F}(0, t_1, T_i) \\ & \left(\exp\left(\int_0^{t_1} dt \int_{t_1}^{T} dx \int_{t_1}^{T_i} dx' \sigma(t, x) D(x - t, x' - t) \sigma(t, x') \right) - 1 \right) \end{aligned} \tag{42}$$

and the minimization of the residual variance of the hedged portfolio proceeds exactly as in the first section. The hedge ratios are found to be given by

$$\Delta = L^T M^{-1} \tag{43}$$

and the minimized variance is again

$$\mathrm{Var}[P(t, T)] - L^T M^{-1} L \tag{44}$$

It is not too difficult to see that both M and L reduce to the results in the first section if $t_1 \to 0$ (with the covariances being scaled by t_1, of course).

One very interesting difference between the instantaneous hedging and finite time hedging is that the result depends on the value of α. In the calculation above, we used the risk-neutral α obtained for the money market numeraire. However, the market does not follow the risk-neutral measure and it would be better to use a value for α estimated for the market for any practical use of this method. This difference is expected since in the very short term only the stochastic term dominates making the drift inconsequential. This, of course, is not the case for the finite time case where the drift becomes important (it is not difficult to see that the importance of the drift grows with the time horizon).

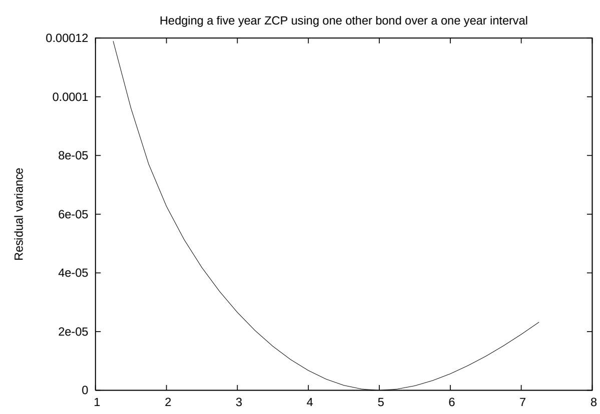

Figure 13: Residual variance when a five year bond is hedged with one other bond (best fit of the constant rigidity field theory model) with a time horizon of one year

6 Semi-empirical Results for Finite Time Hedging

We now present the empirical results for hedging of a bond with other bonds for both the best fit for the constant rigidity field theory model and the fully empirical propagator. The calculation of L and M were again carried out using simple trapezoidal integration and σ was assumed to be purely a function of $\theta = x - t$ so that all the integrals over x were replaced by integrals over θ. The bond to be hedged was chosen to be the five year zero coupon bond and the time horizon t_1 was chosen to be one year.

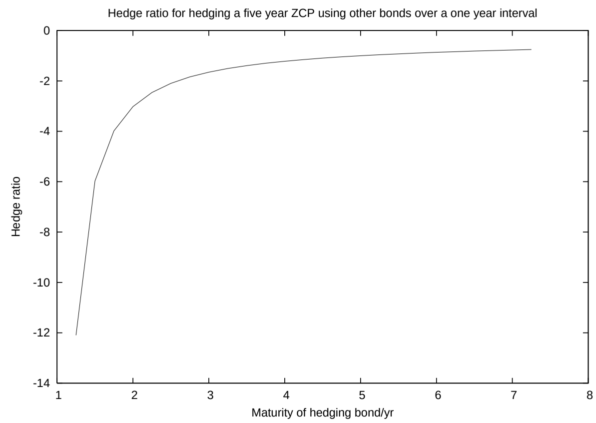

Figure 14: Hedge ratio when a five year bond is hedged with one other bond (best fit of the constant rigidity field theory model) with a time horizon of one year

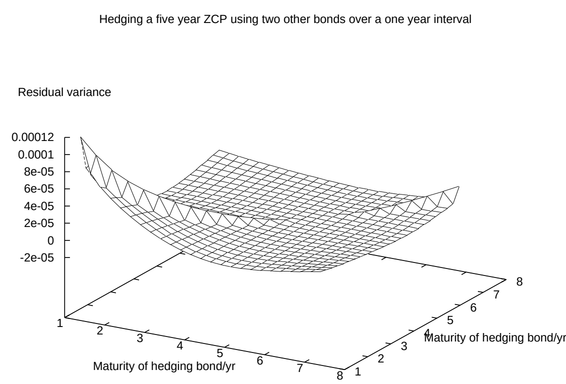

Figure 15: Residual variance when a five year bond is hedged with two other bonds (best fit of the constant rigidity field theory model) with a time horizon of one year

The results for the best fit of the constant rigidity field theory model (see figures 1 and 2) for the residual variance and hedge ratio for hedging with one bond are shown in figures 13 and 14. The residual variance for hedging with two bonds is shown in figure 15.

The results for the fully empirical quadratic fit (see figures 6 and 7) for the residual variance and hedge ratio for hedging with one bond are shown in figures 16 and 17. The residual variance for hedging with two bonds is shown in figure 18.

One interesting result is that the actual residual variance after hedging over a finite time horizon is lesser than naively extrapolating from the infinitesimal hedging result. This seems to be due to the shrinking nature of the domain as the contribution to the variance of the bonds reduces as the time horizon increases. This is very clear if the maturity of the bond is close to the hedging horizon as the volatility of bonds reduces quickly as the time to maturity approaches. Apart from this reduction, the results look very similar to the infinitesimal case. This is probably due to the fact that the volatility is quite small so the nonlinear effects in the covariance matrix (41) are not apparent. If very long time horizons (ten years or more) and long term bonds are considered, the results will probably be quite different. We see by comparing figures 15 and 18 that the better improvement in using more than one bond to hedge when the empirical rather than the field theory model propagator is used is seen to be true in the finite time case as well.

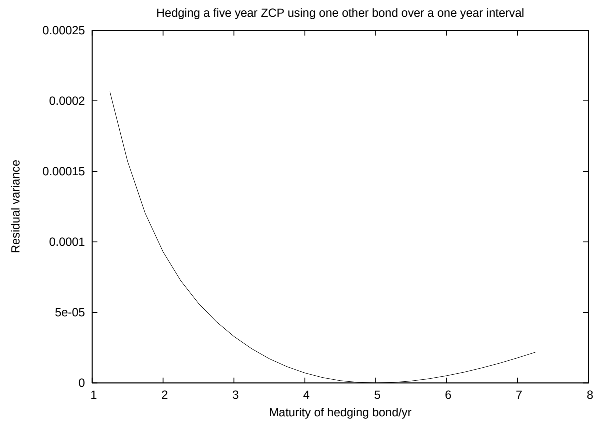

Figure 16: Residual variance when a five year bond is hedged with one other bond (best empirical fit) with a time horizon of one year

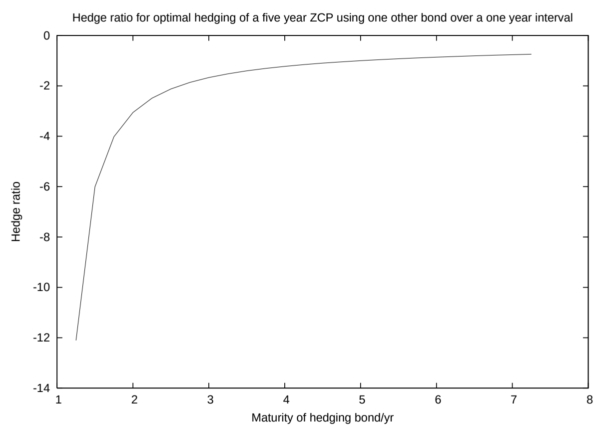

Figure 17: Hedge ratio when a five year bond is hedged with one other bond (best empirical fit) with a time horizon of one year

7 Conclusion

We have shown that the field theory model offers techniques to calculate hedge parameters for fixed income derivatives and provides a framework to answer questions concerning the number and maturity of bonds to include in a hedge portfolio. We have also seen how the field theory model can be used to estimate hedge parameters for finite time as well which is useful in practice. We have used the field theory model calibrated to market data to show that a low dimensional basis provides a reasonably good approximation within the framework of this model. This shows that field theory models address the theoretical dilemmas of finite factor term structure models and offer a practical alternative to finite factor models.

8 Acknowledgements

We would like to thank Prof. Warachka for many insightful and illuminating discussions. We would also like to thank Jean Phillipe-Bouchaud and Science and Finance for kindly providing us with the data for the semi-empirical section of the study.

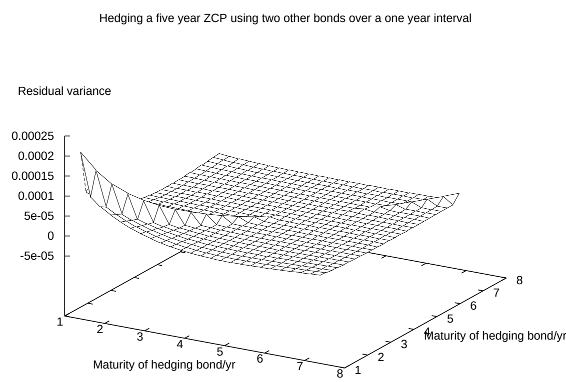

Figure 18: Residual variance when a five year bond is hedged with two other bonds (best empirical fit) with a time horizon of one year